# Steady State Analysis Of Multitone Nonlinear Periodic Circuits In Wavelet Domain

Nick Soveiko and Michel Nakhla

Department of Electronics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada

**Abstract -- A new method for steady state analysis of nonlinear periodic circuits is proposed. The new method is similar to the well known technique of Harmonic Balance, but uses wavelets as basis functions instead of Fourier series. Because of the increased sparsity of the Jacobian matrix, the new method scales linearly with the size of the problem and is well suited for large scale simulations.**

## I. INTRODUCTION

Steady state analysis of nonlinear circuits under periodic excitations remains one of the most computationally challenging tasks in microwave design, particularly for highly nonlinear circuits under multitone excitations. A common approach to numerical solution of this class of problems is to expand the nonlinear differential circuit equation (usually in MNA form) [1], [2]:

$$C\dot{x} + Gx + f(x) + u = 0 \tag{1}$$

in a periodic basis that naturally enforces boundary conditions

$$x(t+\tau) = x(t),\ u(t+\tau) = u(t) \tag{2}$$

This expansion results in a nonlinear algebraic equation

$$\Phi(X) = (\hat{C}D + \hat{G})X + F(X) + U = 0 \tag{3}$$

where

$$X = Tx,\ x = \tilde{T}X,\ U = Tu, \tag{4}$$

matrix $D$ is a representation of the derivative operator in expansion basis $\{\zeta_i\}$:

$$[D_{ij}] = \langle \frac{d}{dt}\zeta_i, \zeta_j \rangle \tag{5}$$

and, finally, $T$ and $\tilde{T}$ are matrices associated with the forward and inverse transform arising from the chosen expansion basis.

Equation (3) is usually solved for $X$ by Newton's iterations, in which case it's Jacobian can be written as

$$J(X) = \frac{\partial \Phi}{\partial X} = \hat{C}D + \hat{G} + T\left[\frac{\partial f}{\partial x}\right]\tilde{T} \tag{6}$$

Traditional approach for such expansion is using Fourier series with current state of the art features being diamond truncation of the base frequency set and frequency mapped Fourier transform ([1], [2]). This method is widely known as Harmonic Balance. HB provides reasonable accuracy and convergence, but exhibits high demands for CPU time and memory storage for large scale simulations.

We are proposing a new method for steady state analysis of periodic nonlinear circuits. Proposed method utilizes orthogonal wavelets [4] as the expansion basis and is significantly faster than Harmonic Balance for highly nonlinear, large scale and broadband circuits, while retaining it's accuracy and convergence properties. We also provide analysis of computational cost of both methods and show that with respect to the highest order of intermodulation products to be retained, traditional approach has computational cost of $O(N^2)$ to $O(N^4)$, while wavelet methods are $O(N)$, which gives them significant advantage in simulations of multitone and highly nonlinear circuits.

## II. WAVELET FORMULATION

Because of the general form matrix formulation, equations (1)-(6) still hold, with minor differences arising from construction of the matrices in (3). Matrix $D$ contains connection coefficients (5) which define projection of the derivative operator onto wavelet space [5]:

$$D = TR\tilde{T} \tag{7}$$

where $R$ is a Toeplitz matrix ([6], p.183) with elements $r_i$ on it's $i$-th diagonal:

$$r_i = \langle \varphi\left(\frac{t}{\tau} - \frac{i}{N_t}\right), \frac{\partial}{\partial t}\varphi\left(\frac{t}{\tau}\right) \rangle, \tag{8}$$

matrices $T$ and $\tilde{T}$ are associated with the forward and inverse wavelet transform, $N_t$ is the number of time points for analysis and $\varphi(t)$ is the scaling function associated with given wavelet. Coefficients $r_i$ are in fact rational numbers and for each type of wavelets can be precomputed symbolically ([5], [8]) and tabulated for future reference. Because we use wavelets that have local support, these



0-7803-7695-1/03/$17.00 

coefficients are nonzero only for small values of $i$. Because of that matrix $R$ is structurally sparse. Transform matrices $T$ and $\tilde{T}$ are block diagonal, with blocks constructed from filter coefficients associated with the particular wavelet and thus are also structurally sparse and bandlimited [7]. This means that matrix $D$ is also bandlimited and sparse with $O(N_t)$ nonzero entries (Fig. 1). This also means that $T(\partial f/\partial x)\tilde{T}$ in (6) is a sparse bandlimited matrix with $O(N_t)$ nonzero entries (Fig. 2). Periodic boundary conditions are enforced by using periodized wavelets [8] which preserve sparsity of the matrices $D$, $T$ and $\tilde{T}$ and that of Jacobian (6) in general.

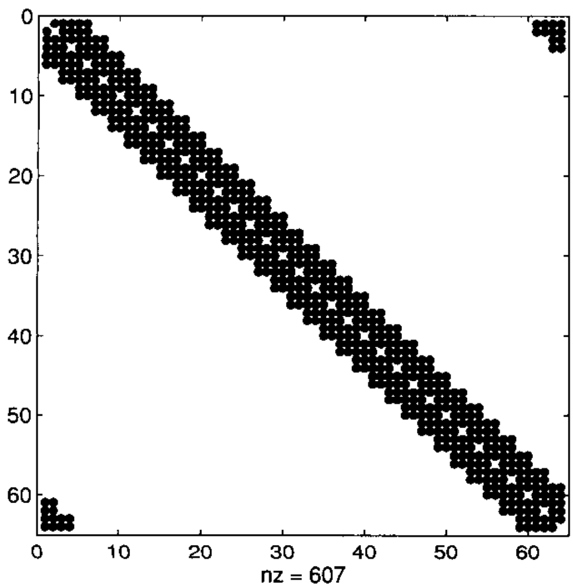


Fig. 1 Sparsity pattern for matrix $D$ expanded in a basis of periodic orthogonal local support wavelets.

## III. COMPUTATIONAL COST ANALYSIS

Let us consider equation (1) in scalar form. Provided orthogonal Fourier transform is used, matrices $T$ and $\tilde{T}$ in (4) are square and dense ([1]). Jacobian (6) also becomes a dense matrix because of the $T(\partial f/\partial x)\tilde{T}$ component. If we denote order of expansion as $N_t$, Jacobian is a dense $N_t \times N_t$ matrix that has $O(N_t^2)$ nonzero elements.

We can generalize this to a vector case. Matrices in (3) and (6) have $N_t \times N_t$ block structure with each nonzero block corresponding to one nonzero entry in circuit equation matrices (1). Density of Jacobian (6) in this case is dominated by dense blocks corresponding to nonlinear elements in the circuit (Fig. 3). Only the size of these blocks changes with the order of expansion. Overall density of the

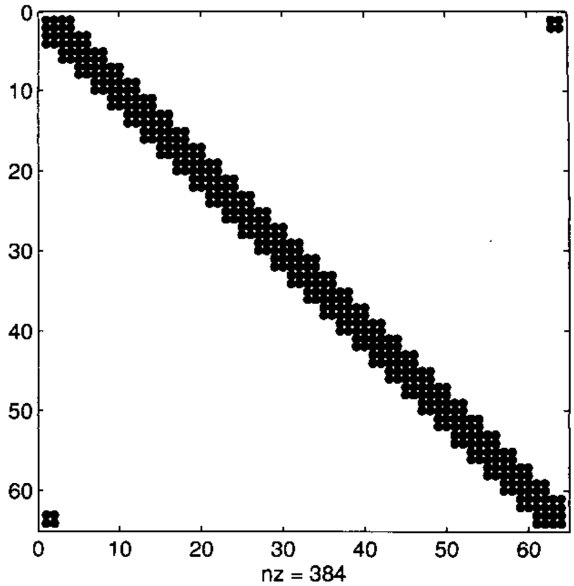


Fig. 2 Sparsity pattern for the $T(\partial f/\partial x)\tilde{T}$ component of Jacobian expanded in a basis of periodic orthogonal local support wavelets.

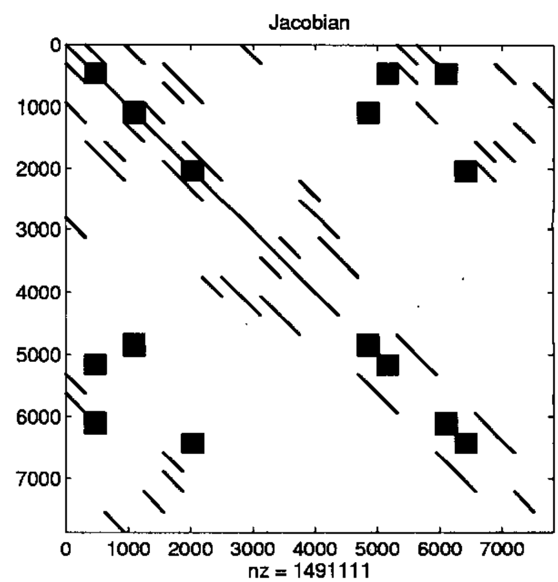


Fig. 3 Example of sparsity pattern for Jacobian resulting from Fourier series expansion of the MNA equations. Dense blocks account for 98.6% of nonzero elements.

Jacobian in this case is $N_{NZ} \cong O(K \cdot N_t^2)$, where $K$ is constant for a given circuit and therefore

$$N_{NZ} \cong O(N_t^2) \tag{9}$$

like in scalar case. In it's turn, order of expansion is linearly proportional to the number of frequencies in truncated set:

$$N_t = 2N_f + 1 \tag{10}$$

In general case, for multitone analysis with $S$ tones, box or diamond truncation and highest order of IM products retained in analysis equal to $N_H$, number of frequencies in truncated set is proportional to the volume of $S$-dimensional cube $N_f \cong O(N_H^S)$ ([1], p.245). However, for a practically important case of intermodulation analysis with equidistant tone frequencies (e.g. 900, 910, 920, ... MHz), frequencies of the new IM products due to increase in $N_H$ often coincide with already existing in the grid, thus reducing cost to

$$N_f \cong O(S \cdot N_H^2) \tag{11}$$

Combining (9), (10) and (11), we conclude that for multitone Harmonic Balance computational cost in terms of the number of nonzero elements in Jacobian is equal to at least $N_{NZ} \cong O(N_H^4)$ .

Similarly to HB expansion, estimations given in previous section for wavelet expansion can be generalized to include circuit equations (1), where each nonzero element after expansion becomes an $N_t \times N_t$ sparse block, each having $O(N_t)$ nonzero entries (see Fig. 1 and Fig. 2). Total number of nonzero entries in wavelet Jacobian becomes

$$N_{NZ} \cong O(N_t) \tag{12}$$

with $N_t = 2N_f$ because of the sampling theorem.

With wavelets we use trivial truncation that produces an equidistant uniform frequency grid spanning harmonics and IM components up to required $N_H$. This, however, is quite a beneficial trade-off as this scheme produces frequency grid with

$$N_f = O\left(\frac{N_H}{\Delta\hat{f}}\right) \tag{13}$$

components, where $\Delta\hat{f}$ is the relative density if the frequency grid (e.g. for base frequencies 99 and 100 MHz $\Delta\hat{f} = 1\%$). Combining (12) and (13) we conclude that computational cost of wavelet expansion is

$$N_{NZ} = O(N_H) \tag{14}$$

and despite the primitive truncation schemes, with increase in $N_H$ and $S$ wavelet methods very quickly gain significant advantages in computational cost (Fig. 4).

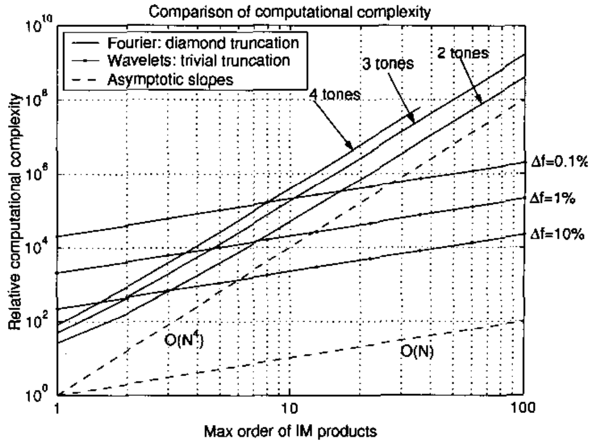


Fig. 4. Comparison of computational complexity in terms of the number of nonzero elements in the Jacobian.

## IV. NUMERICAL RESULTS

All simulations referred in this section were performed in Matlab 6.5.0 (R13), running on a SUN Blade-1000 workstation with 900 MHz UltraSPARC-III CPU and 5 GB of physical RAM. In both examples diamond truncation was used for Harmonic Balance and trivial truncation for wavelet expansion. Daubechies wavelets of second order were used for wavelet expansion in both examples. Both methods produced essentially identical results in terms of accuracy and convergence (not shown here due to space constraints).

### A. Cascode amplifier

In the first example a 900 MHz cascode LNA was considered. The amplifier consists of 2 BJTs with DC bias and impedance matching networks. Total size of MNA equations for this example is 25. Two tones with frequencies of 900 and 910 MHz and equal amplitudes were used for the input. In each case (HB and wavelets) third order in-band intermodulation products with frequencies of 920 and 930 MHz were computed with $N_H$ ranging from 5 to 22 (maximum value for HB given software implementation and available memory). Number of nonzero elements in the Jacobian was recorded and is shown in Fig. 5.

Results exhibit good correspondence with asymptotical computational complexity estimates derived in the previous section (Fig. 4). Of particular interest is the intersection of HB and wavelet plots which occurs between $N_H = 9$ and $N_H = 10$ (note the same region in Fig. 4 for $\Delta\hat{f} = 1\%$).

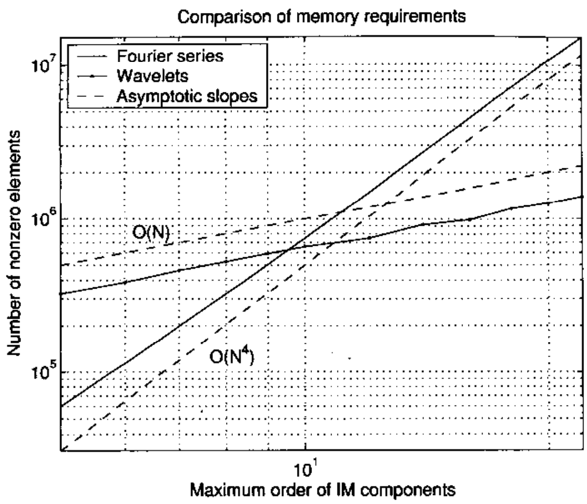


Fig. 5. Number of nonzero elements in the Jacobian for cascode amplifier circuit.

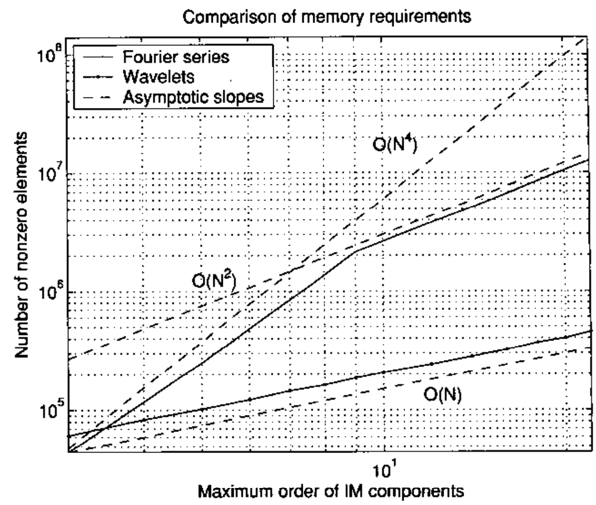


Fig. 6. Number of nonzero elements in the Jacobian for Gilbert cell mixer circuit.

*B. Gilbert cell mixer*

The second example involves a BJT Gilbert cell mixer circuit that consists of 9 transistors (including 3 as current sources), DC bias and impedance matching networks. The mixer was configured for down conversion with LO input at 1 GHz, RF input at 900 MHz and IF output at 100 MHz. Total size of MNA equations for this circuit is 37. In each case conversion gain was computed as a function of RF power while keeping LO power at +1 dBm. Simulations were performed for $N_H$ ranging from 3 to 22 (again, limited by memory requirements for HB simulations). For each $N_H$ maximum number of nonzero elements in the Jacobian was recorded and is shown in Fig. 6.

The plot is in a good agreement with theoretical estimations (Fig. 4). Fig. 6 shows that before $N_H = 9$ HB memory requirements grow slightly slower than $O(N_H^4)$ which can be explained by the fact that $T(\partial f/\partial x)\tilde{T}$ components in (6) do have some additional sparsity within mostly dense blocks. After $N_H = 9$ memory requirements grow as $O(N_H^2)$ which is due to the quick saturation of the diamond truncation grid with base frequencies of 900 and 1000 MHz, after which grid becomes equidistant and $N_f$ grows essentially as $O(N_H)$ as opposed to (11). However, these minor variations from the trend do not affect the fact that wavelet expansion has stable computational cost of $O(N_H)$ and becomes preferable at around $N_H = 4$ with gain exceeding order of magnitude by the time $N_H$ reaches 10.

## V. Concluding Remarks

In this paper we have shown that proposed wavelet method for steady state analysis of nonlinear periodic circuits has $O(N_H)$ computational cost as opposed to $O(N_H^4)$ for traditional Harmonic Balance. This makes wavelet techniques particularly attractive for simulations of highly nonlinear, large scale and broadband circuits.

## References


[1] Paulo J.C. Rodrigues, *Computer-aided analysis of nonlinear microwave circuits*, Norwood: Artech House, 1998.

[2] Kenneth S. Kundert, Jacob K. White and Alberto Sangiovanni-Vincentelli, *Steady-state methods for simulating analog and microwave circuits*, Boston: Kluwer Academic Publishers, 1990.

[3] Michel S. Nakhla, Jiri Vlach, "A piecewise harmonic balance technique for determination of periodic response of nonlinear systems", *IEEE Trans. on Circuits and Systems*, vol. CAS-23, no. 2, pp. 85-91, February 1975.

[4] Ingrid Daubechies, *Ten Lectures on Wavelets*, Philadelphia: SIAM, 1992.

[5] G. Beylkin, "On the representation of operators in bases of compactly supported wavelets", *SIAM J. on Numerical Analysis*, Vol. 6, No. 6, pp. 1716-1740, June 1992.

[6] Gene H. Golub, Charles F. Van Loan, *Matrix computations*, Baltimore: The John Hopkins University Press, 1989.

[7] Yves Nievergelt, *Wavelets Made Easy*, Boston: Birkhauser, 1999.

[8] Juan Mario Restrepo, Gary K. Leafy, "Inner Product Computations Using Periodized Daubechies Wavelets", *International Journal of Numerical Methods in Engineering*, 40, pp. 3557-3578, 1997.